Heat Transport in Quantum Systems:

An Approch Using Projection Operator Techniques

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Outline

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Heat Transport

Local Equilibrium

Heat Conduction Experiment
(Deutsches Museum München)

- no global temperature
- local temperature
- temperature gradient
- constant heat current

➜ Normal Transport

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Jean Baptiste Joseph Fourier
(1768 - 1830)

Fourier's Law

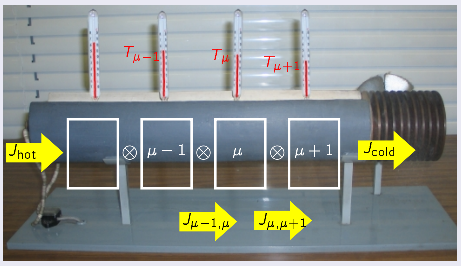
linear connection between
temperature gradient ∇T and
heat current $\boldsymbol{J}$

$$\boldsymbol{J} = -\kappa \nabla T$$

➜ heat conductivity κ

Heat Conductivity

Discrete Version Fourier's Law



$$\begin{aligned} J_{\mu,\mu+1} &= -\kappa(T_{\mu+1} - T_\mu) \\ &= \kappa\,\Delta T_{\mu,\mu+1} \end{aligned}$$

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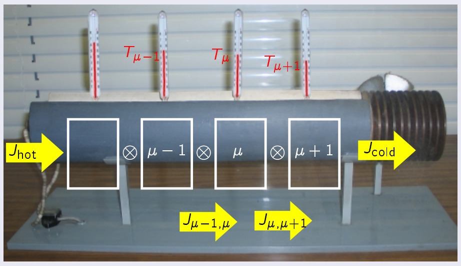


$$\begin{aligned} J_{\mu,\mu+1} &= -\kappa(T_{\mu+1} - T_\mu) \\ &= \kappa\, \Delta T_{\mu,\mu+1} \end{aligned}$$

Conditions for Local Equilibrium

$$J_{\text{hot}} = J_{\mu-1,\mu} = J_{\mu,\mu+1} = J_{\text{cold}} = J$$

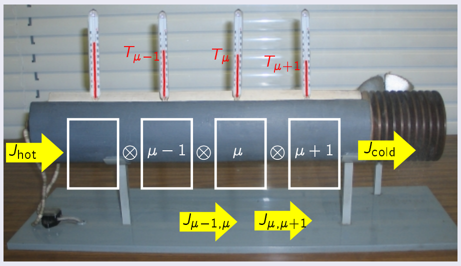
for κ to be a material constant

$$\Delta T_{\mu-1,\mu} = \Delta T_{\mu,\mu+1} = \Delta T$$

except some differences at the contacts

Heat Conductivity

Discrete Version Fourier's Law



$$J_{\mu,\mu+1} = -\kappa(T_{\mu+1} - T_\mu)$$
$$= \kappa\, \Delta T_{\mu,\mu+1}$$

Definition of the Heat Conductivity

$$\kappa = \frac{J}{\Delta T}$$

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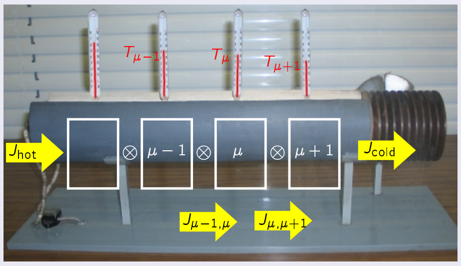

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except some differences at the contacts

Normal and Ballistic Transport

$$\Delta T \neq 0 \quad \Rightarrow \quad \text{finite} \quad \kappa$$

➜ normal transport

$$\Delta T = 0 \quad \Rightarrow \quad \kappa \to \infty$$

➜ ballistic transport

Microscopic Foundation

Peter J. W. Debye (1884-1966)

particle current:
electrons

- mean velocity $\overline{v}$
- mean free path $\overline{l}$

$\kappa \propto c\,\overline{v}\,\overline{l}$

(c heat capacity)

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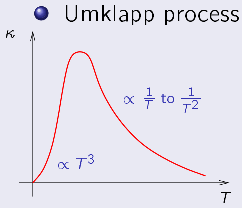
quasi-particle current:
phonons

- Boltzmann equation
- Umklapp process



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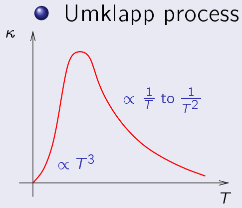
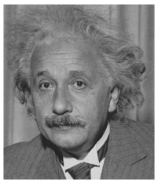


Ryogo Kubo (1920-1995)

- linear response theory (for electrical transport)
- Kubo formula ➜ current current autocorrelation
- ad hoc transferred from electrical to heat transport

Connection between Diffusion and Relaxation



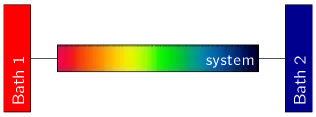
Albert Einstein (1879-1955)

Marian Ritter von Smolan Smoluchowski (1872-1917)

Diffusion

The diffusion constant in a *local equilibrium steady state* (bath scenario) is equivalent to the relaxation constant from a far from equilibrium state into the global equilibrium (decay scenario).

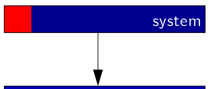
Bath Scenario



stationary local equilibrium state

➜ heat conductivity

Decay Scenario



initial state

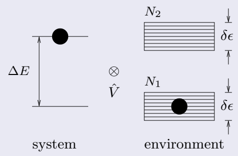
final global equilibrium

➜ relaxation constant

Outline

Model System

Spin coupled to finite environments



System parameters:

λ	coupling strength (=0.0005)
$\delta\epsilon$	band width (=0.5)
$N1$, $N2$	number of levels (=500)
ΔE	local splitting (=10)

Model System

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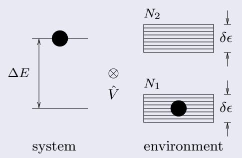

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System dynamics

Hamiltonian

$$\hat{H} = \hat{H}_{\mathrm{sys}} + \hat{H}_{\mathrm{env}} + \lambda \hat{V}$$

Schrödinger dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = -\mathrm{i}\hat{H} |\psi(t)\rangle$$

Reduced density operator

$$\hat{\rho}_{\mathrm{S}} = \mathrm{Tr}_{\mathrm{E}} \{ |\psi(t)\rangle\langle\psi(t)| \}$$

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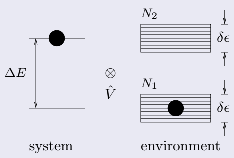

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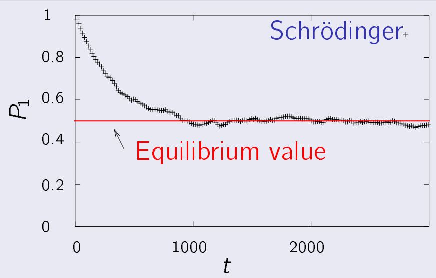
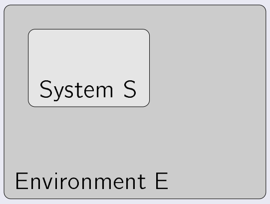
Dynamical Investigation



Reduced Description

Closed System: System and Environment



Schrödinger pure state dynamics $|\psi\rangle$ or $\hat{\rho} = |\psi\rangle\langle\psi|$ according to Liouville von Neumann Equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\mathrm{i}[\hat{H}, \hat{\rho}] = \mathcal{L}\hat{\rho}$$

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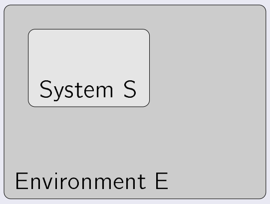

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Reduced dynamics for the system S

$$\hat{\rho}_{\mathrm{S}} = \mathrm{Tr}_{\mathrm{E}}\left\{\hat{\rho}\right\} \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}_{\mathrm{E}}\left\{\hat{\rho}\right\} = \frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_{\mathrm{S}} = \mathrm{Tr}_{\mathrm{E}}\left\{\mathcal{L}\hat{\rho}\right\} \quad \rightarrow \quad \text{not closed}$$

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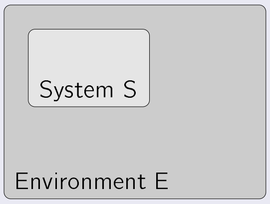

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Approaches to a closed equation

- Hilbert Space Average Method (HAM)
- Projection Operator Technique: Nakajima-Zwanzig (NZ)
- Projection Operator Technique: Time Convolutionless (TCL)

Quantum Master Equation (TCL)

Projection operator to relevant (irrelevant) part of the system

Superoperator: $\mathcal{P}\hat{\rho}$ = relevant part $\mathcal{Q}\hat{\rho} = \hat{\rho} - \mathcal{P}\hat{\rho}$ = irrelevant part

standard choice: $\mathcal{P}\hat{\rho} = \mathrm{Tr}_{\mathrm{E}}\{\hat{\rho}\} \otimes \hat{\rho}_{\mathrm{E}}$

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Irrelevant Part

LvN equation: $\dfrac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = \mathcal{L}(t)\hat{\rho}(t)$

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Unit operator: $\mathcal{P} + \mathcal{Q} = 1$

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Formal solution

$$\mathcal{Q}\hat{\rho}(t) = \int_0^t \mathrm{d}s\, \mathcal{G}(t,s)\mathcal{Q}\mathcal{L}(s)\mathcal{P}\hat{\rho}(s)$$

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Time Locallity

Replace the density $\hat{\rho}(s)$ by

$\hat{\rho}(s) = G(s,t)\hat{\rho}(t)$

Backwards propagator ➜ full system

Formal solution

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$$[1 - \mathcal{S}(t)]\mathcal{Q}\hat{\rho}(t) = \mathcal{S}(t)\mathcal{P}\hat{\rho}(t)$$

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Backwards propagator ➜ full system

Relevant Part

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}\hat{\rho}(t) - \mathcal{P}\mathcal{L}(t)\mathcal{P}\hat{\rho}(t) = \mathcal{P}\mathcal{L}(t)\mathcal{Q}\hat{\rho}(t)$$

Formal solution

$$\mathcal{Q}\hat{\rho}(t) = \left[1 - \mathcal{S}(t)\right]^{-1}\mathcal{S}(t)\mathcal{P}\hat{\rho}(t)$$

Quantum Master Equation (TCL)

Exact time local equation (see Breuer, "*Open Quantum Systems*")

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}\hat{\rho}(t) = \mathcal{P}\mathcal{L}(t)\mathcal{P}\hat{\rho}(t) + \mathcal{P}\mathcal{L}(t)\mathcal{Q}\hat{\rho}(t)\,, \qquad \mathcal{Q}\hat{\rho}(t) = [1-\mathcal{S}(t)]^{-1}\mathcal{S}(t)\mathcal{P}\hat{\rho}(t)$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}\hat{\rho}(t) = \underbrace{\mathcal{K}(t)}_{\text{TCL generator}}\ \mathcal{P}\hat{\rho}(t) \qquad \text{complicated but exact}$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}\hat{\rho}(t) = \underbrace{\mathcal{K}(t)}_{\text{TCL generator}}\mathcal{P}\hat{\rho}(t) \qquad \text{complicated but exact}$$

Series expansion of the TCL generator

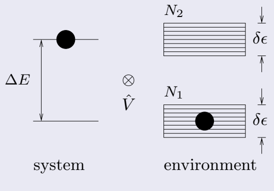
$$\mathcal{K}(t) = \sum_{i=1}^{\infty} \lambda^i\,\mathcal{K}_i(t) \qquad \lambda = \text{coupling strength}$$

Born Approximation ➜ TCL Master Equation:

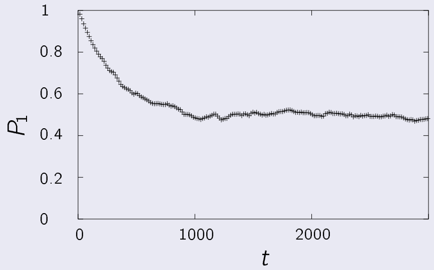
- truncate at 2nd order TCL2: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}\hat{\rho}(t) = \lambda^2\mathcal{K}_2(t)\mathcal{P}\hat{\rho}(t)$
- all odd orders vanish
- next higher is 4th order TCL4: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}\hat{\rho}(t) = [\lambda^2\mathcal{K}_2(t) + \lambda^4\mathcal{K}_4(t)]\mathcal{P}\hat{\rho}(t)$

Application to the Model System

Comparison of TCL and the exact dynamics



$\lambda = 0.0005$, $\delta\epsilon = 0.5$,
$N1 = N2 = N = 500$



see Breuer et al. PRE 73, 016139 (2006)

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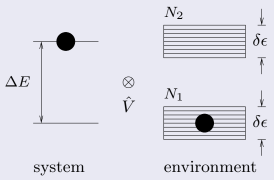

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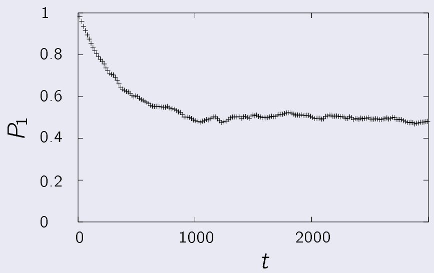

Standard Projector

$$\mathcal{P}\hat{\rho} = \mathrm{Tr_E}\left\{\hat{\rho}\right\} \otimes \hat{\rho}_\mathrm{E}$$

see Breuer et al. PRE 73, 016139 (2006)

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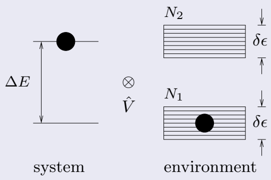

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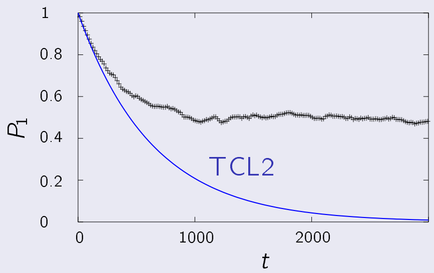

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Solution of the QME in TCL2

$$P_1(t) = P_1(0)\mathrm{e}^{-\gamma t}$$

with the decay rate $\gamma = \frac{2\pi\lambda^2 N}{\delta\epsilon}$

see Breuer et al. PRE 73, 016139 (2006)

Application to the Model System

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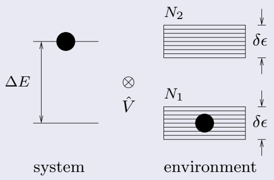

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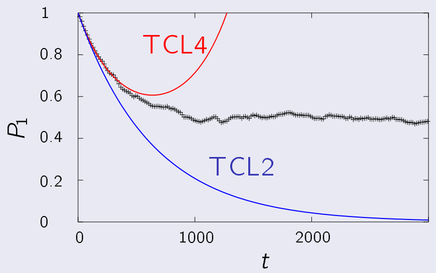

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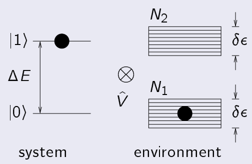
Problem

- TCL2 converges to the wrong equilibrium value
- all higher orders of the TCL expansion are divergent

see Breuer et al. PRE 73, 016139 (2006)

A Correlated Projector for TCL

What goes wrong in the system?



- Energy conservation in the whole system

Breuer PRA 75, 022103 (2007),
Breuer et al. PRE 73, 016139 (2006)

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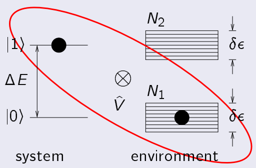

- Energy conservation in the whole system
- Initial state:
  $|1\rangle \otimes |\text{bath somewhere in lower band}\rangle$

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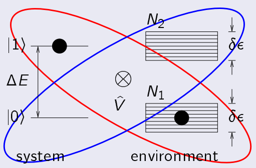

- Energy conservation in the whole system
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- Dynamics:
  $|1\rangle \otimes |$bath somewhere in lower band$\rangle +$
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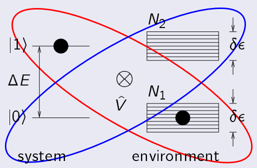

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$\hat{\rho}_{\mathrm{S}} = \mathrm{Tr}_{\mathrm{E}}\{\hat{\rho}\}$ dynamic variable

Breuer PRA 75, 022103 (2007),
Breuer et al. PRE 73, 016139 (2006)

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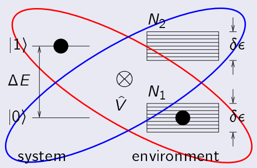

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Band Projectors

$$\hat{\Pi}_i = \sum_{n_i=1}^{N_i} |n_i\rangle\langle n_i|$$

$|n_i\rangle$ state in the ith band

Breuer PRA 75, 022103 (2007),
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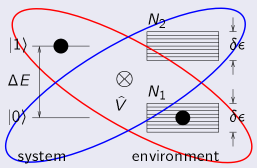

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$|n_i\rangle$ state in the ith band

Correlated Projection Operators

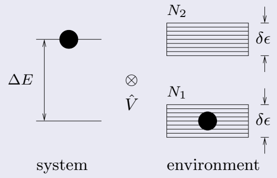
$$\mathcal{P}\hat{\rho} = \sum_i \mathrm{Tr}_{\mathrm{E}}\left\{\hat{\Pi}_i\,\hat{\rho}\right\} \otimes \frac{1}{N_i}\hat{\Pi}_i$$
$$= \sum_i \hat{\rho}_i \otimes \frac{1}{N_i}\hat{\Pi}_i$$

$\hat{\rho}_i$ dynamic variable of the system

Breuer PRA 75, 022103 (2007),
Breuer et al. PRE 73, 016139 (2006)

A Correlated Projector for TCL

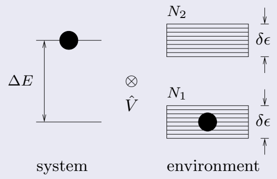
Correlated TCL Master Equation



$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_1 = \gamma(t)\left(\hat{\sigma}_+\hat{\rho}_2\hat{\sigma}_- - \frac{1}{2}[\hat{\sigma}_+\hat{\sigma}_-, \hat{\rho}_1]_+\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_2 = \gamma(t)\left(\hat{\sigma}_-\hat{\rho}_1\hat{\sigma}_+ - \frac{1}{2}[\hat{\sigma}_-\hat{\sigma}_+, \hat{\rho}_2]_+\right)$$

A Correlated Projector for TCL

Correlated TCL Master Equation



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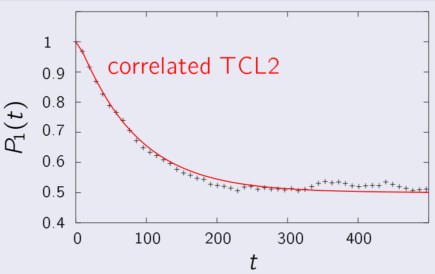
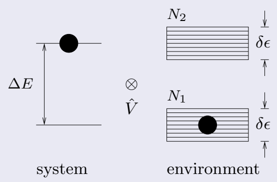
$\lambda = 0.001$



$P_1(t) = 0.5 + 0.5\mathrm{e}^{-2\gamma t}$, $\gamma = \frac{2\pi\lambda^2 N}{\delta\epsilon}$

A Correlated Projector for TCL

Correlated TCL Master Equation



$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_1 = \gamma(t)\big(\hat{\sigma}_+\hat{\rho}_2\hat{\sigma}_- - \frac{1}{2}[\hat{\sigma}_+\hat{\sigma}_-,\hat{\rho}_1]_+\big)$$
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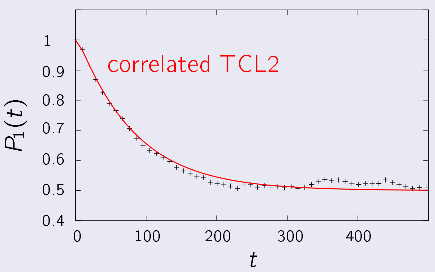

$P_1(t) = 0.5 + 0.5\mathrm{e}^{-2\gamma t}$, $\gamma = \frac{2\pi\lambda^2 N}{\delta\epsilon}$

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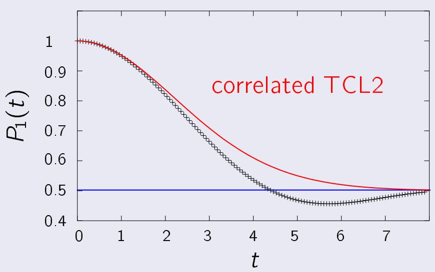
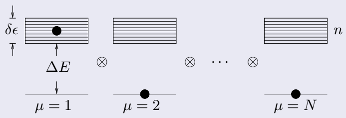


$P_1(t) = 0.5 + 0.5\mathrm{e}^{\Gamma(t)}$

Outline

The Hamiltonian Model

Hamiltonian: Design Model



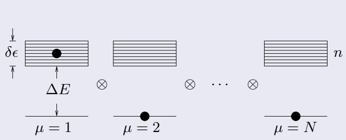
$$\hat{H} = \sum_{\mu=1}^{N} \hat{H}_{\text{loc}}(\mu) + \lambda \sum_{\mu=1}^{N-1} \hat{V}(\mu, \mu+1)$$

$\hat{H}_{\text{loc}}$	local Hamiltonian
$\hat{V}$	random interaction
λ	coupling strength

Michel et al. PRL 95, 180602 (2005), PRE 73, 016101 (2006)

The Hamiltonian Model

Hamiltonian: Design Model



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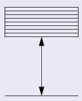
Observed quantity: probability to find μth subsystem excited

- Energy in a subunit ➜ $\Delta E \cdot$ excitation probability P_μ



Subunit μ

Michel et al. PRL 95, 180602 (2005), PRE 73, 016101 (2006)

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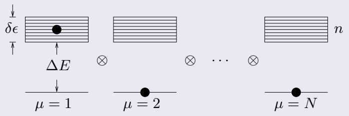

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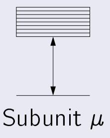

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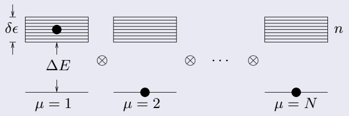

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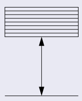
- Probability $P_\mu(t) \equiv \langle \psi(t)|\hat{P}_\mu|\psi(t)\rangle$



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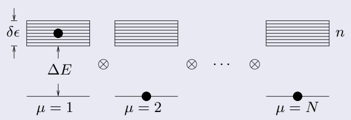
Reduced and Full Dynamics

Model System



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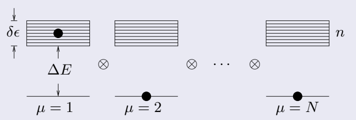
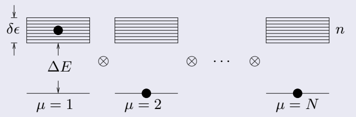


Projection Operator

$$\mathcal{P}\hat{\rho} = \sum_{\mu} \mathrm{Tr}\{\hat{P}_{\mu}\hat{\rho}\}\frac{1}{n}\hat{P}_{\mu} \equiv \sum_{\mu} P_{\mu}\frac{1}{n}\hat{P}_{\mu}$$

Reduced and Full Dynamics

Model System



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TCL

Liouville von Neumann Equation

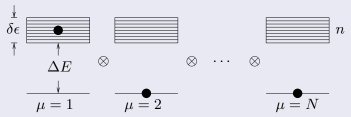
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = \mathcal{L}\hat{\rho}$$

TCL master equation in 2nd order

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}\hat{\rho} = \mathcal{K}_2(t)\mathcal{P}\hat{\rho}$$

Reduced and Full Dynamics

Model System



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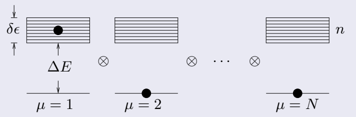
Reduced Dynamics for Probabilities

$$\frac{\mathrm{d}}{\mathrm{d}t}P_1 = -\kappa\Big(P_1 - P_2\Big)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}P_{\mu} = -\kappa\Big(2P_{\mu} - P_{\mu+1} - P_{\mu-1}\Big)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}P_N = -\kappa\Big(P_N - P_{N-1}\Big)$$

- Discrete Diffusion Equation
- Decay constant $\kappa = 2\pi\lambda^2\frac{n}{\delta\epsilon}$

Reduced and Full Dynamics

Model System



Projection Operator

$$\mathcal{P}\hat{\rho} = \sum_{\mu} \mathrm{Tr}\{\hat{P}_{\mu}\hat{\rho}\}\frac{1}{n}\hat{P}_{\mu} \equiv \sum_{\mu} P_{\mu}\frac{1}{n}\hat{P}_{\mu}$$

TCL

Liouville von Neumann Equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = \mathcal{L}\hat{\rho}$$

TCL master equation in 2nd order

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}\hat{\rho} = \mathcal{K}_2(t)\mathcal{P}\hat{\rho}$$

Reduced Dynamics for Probabilities

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}P_1 &= -\kappa\Big(P_1 - P_2\Big)\\
\frac{\mathrm{d}}{\mathrm{d}t}P_{\mu} &= -\kappa\Big(2P_{\mu} - P_{\mu+1} - P_{\mu-1}\Big)\\
\frac{\mathrm{d}}{\mathrm{d}t}P_N &= -\kappa\Big(P_N - P_{N-1}\Big)
\end{aligned}$$

- Discrete Diffusion Equation
- Decay constant $\kappa = 2\pi\lambda^2\frac{n}{\delta\epsilon}$

Numerical Effort

Full Schrödinger equation
in one excitation subspace:

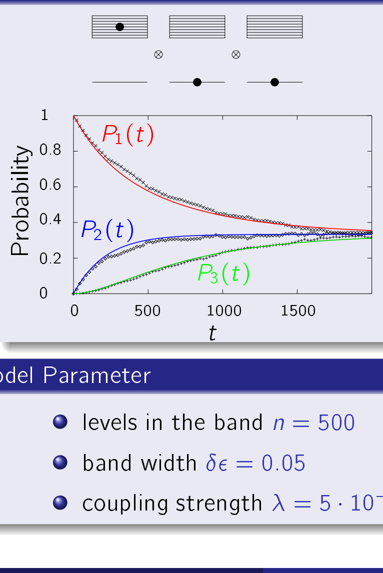
$n \cdot N$ differential equations

TCL master equation:

N differential equations

Comparison of Full and Reduced Dynamics

N=3



Model Parameter

- levels in the band $n = 500$
- band width $\delta\epsilon = 0.05$
- coupling strength $\lambda = 5 \cdot 10^{-5}$

Comparison of Full and Reduced Dynamics

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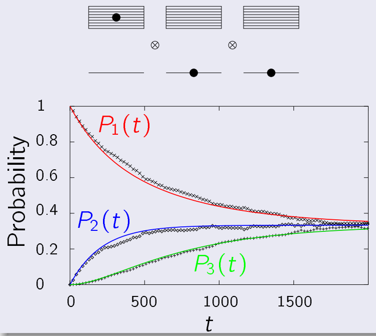
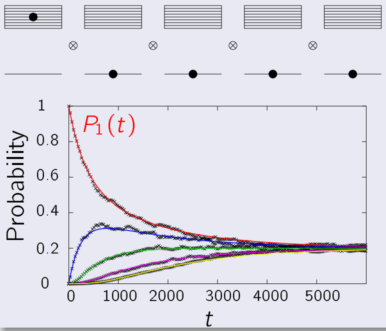


N=5



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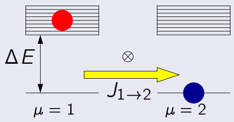
Energy Transport

HAM rate equation



rate equation
➜ exponential decay

$$\frac{\mathrm{d}P_1}{\mathrm{d}t} = -\kappa\left(P_1 - P_2\right)$$
$$\frac{\mathrm{d}P_2}{\mathrm{d}t} = -\kappa\left(P_2 - P_1\right)$$

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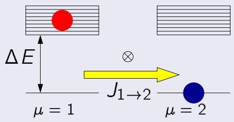

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Energy Current

defined as the change of internal energy U_μ in the two subunits

$$J_{1\to 2} = \frac{1}{2}\left(\frac{dU_2}{dt} - \frac{dU_1}{dt}\right)$$

internal energy: $U_\mu = \Delta E P_\mu$

$$J_{1\to 2} = \frac{\Delta E}{2}\left(\frac{dP_2}{dt} - \frac{dP_1}{dt}\right)$$

and with rate equation

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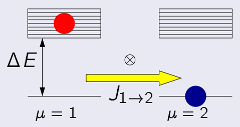

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and with rate equation

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Energy Fourier's law

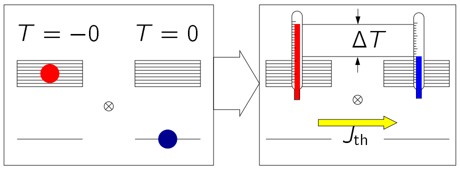
energy current $\propto$ energy gradient

$$J_{1\to 2} = -\kappa(U_2 - U_1)$$

Energy Conduction Coefficient:

$$\kappa = 2\pi\lambda^2 \frac{n}{\delta\epsilon}$$

Heat Transport



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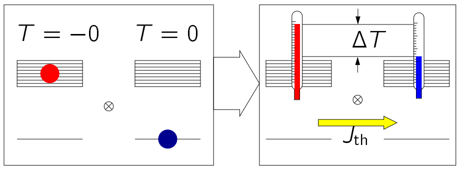
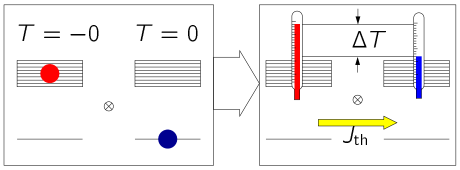


Fourier's Law

$$J_{\mathrm{th}} = -\kappa_{\mathrm{th}} \Delta T$$

Heat Transport



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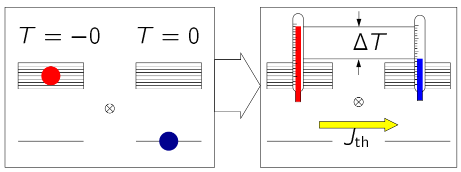
$$\kappa_{\mathrm{th}} = \kappa c$$

κ energy diffusion constant
c heat capacity of subunit

Heat Transport



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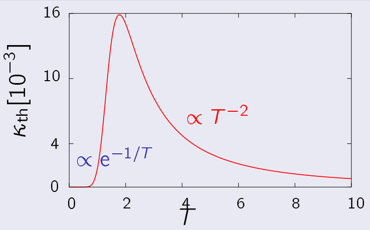


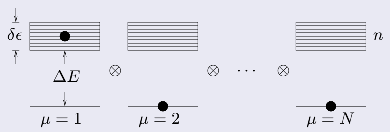
Standard from solid state physics (Peierls)

- small temperatures $\propto T^3$
- high temperatures $\propto T^{-1}$-T^{-2}

Outline

Beyond the Design Model

Design Model



Assets

- local energy and temperature
- energy and heat current

Beyond the Design Model

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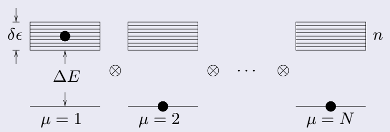

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Drawbacks

- not a real microscopic model
- large gap is not typical

Beyond the Design Model

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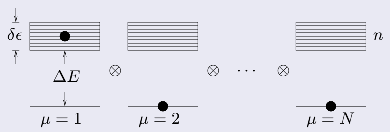

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General Requirements for a Heat Conduction Model

organize the microscopic system as a "net-structure" of subunits

- local high state density
- weakly interacting subunits

Beyond the Design Model

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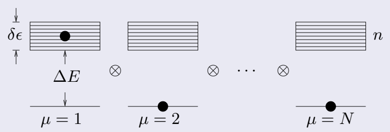

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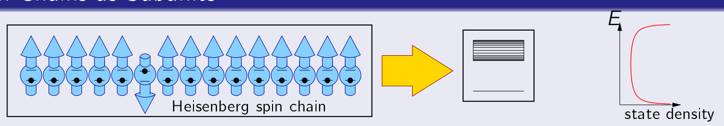
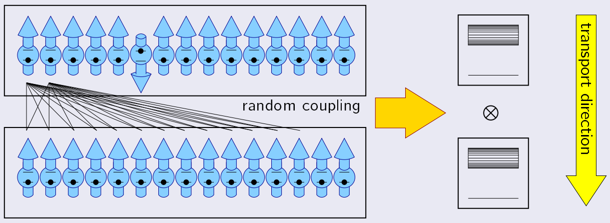
Spin Chains as Subunits



Spin Chain Model

Coupling Model



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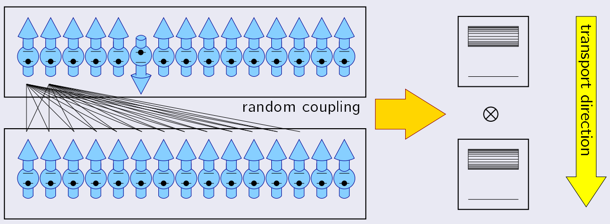
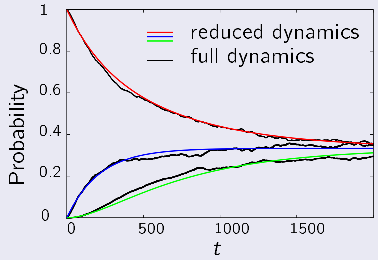


Comparison of Theory and Schrödinger Dynamics



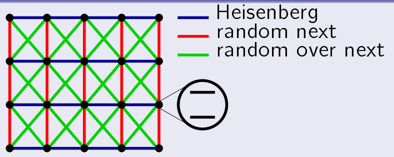
- normal transport behavior
- heat conductivity as before

$$\kappa = 2\pi\lambda^2 \frac{n}{\delta\epsilon}$$

n number of spins

Anisotropic Model

Coupling Model



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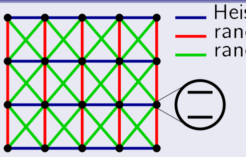
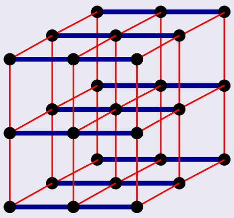


3D Model



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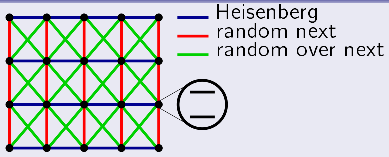

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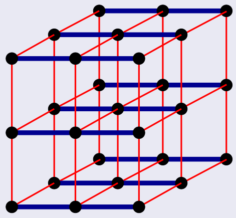

Hamiltonian

$$\hat{H} = \hat{H}_Z + \lambda_{\text{Heis}} \hat{H}_{\text{Heis}} + \lambda \hat{H}_{\text{rand1}} + \lambda \hat{H}_{\text{rand2}}$$

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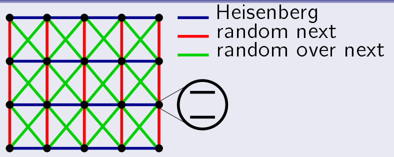

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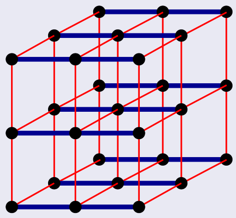

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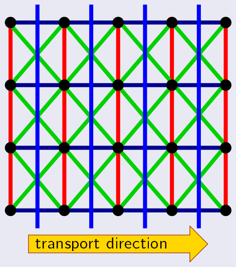
Parallel to Chains



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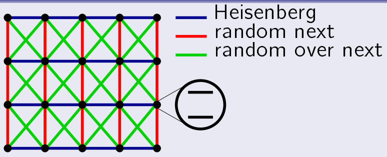

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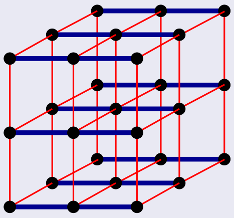

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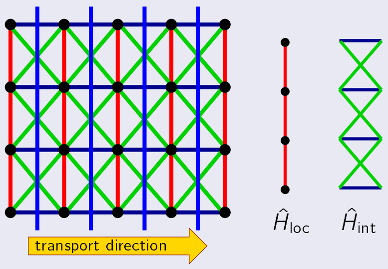

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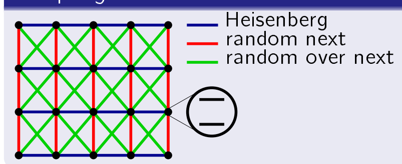

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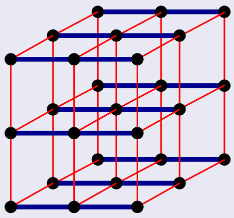

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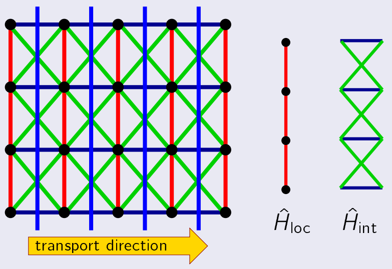
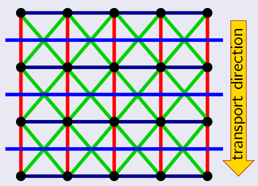


Perpendicular to Chains



Anisotropic Model

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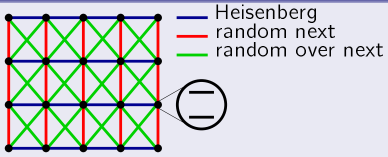

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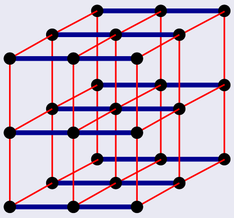

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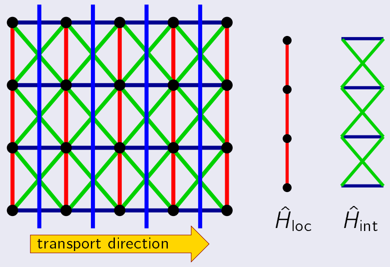

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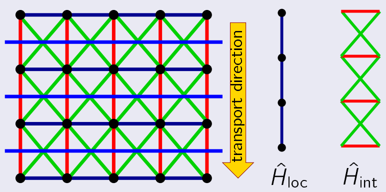
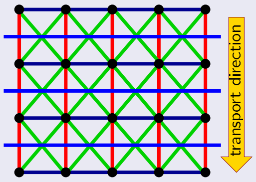


Perpendicular to Chains

Normal Transport



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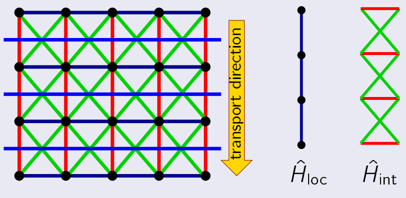

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Perpendicular to Chains

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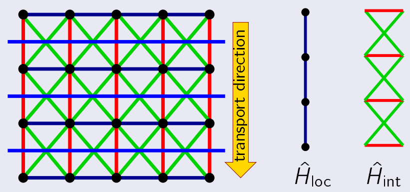

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Projector

$$\mathcal{P}\hat{\rho} = \sum_{\mu} P_{\mu} \frac{1}{n} \hat{P}_{\mu}$$

Perpendicular to Chains

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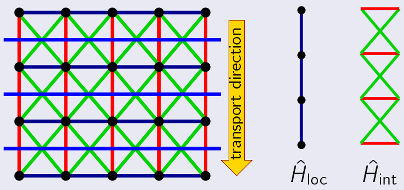

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TCL master equation

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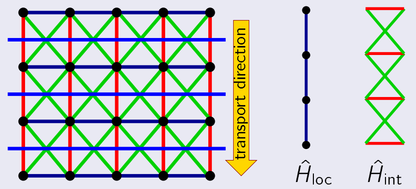

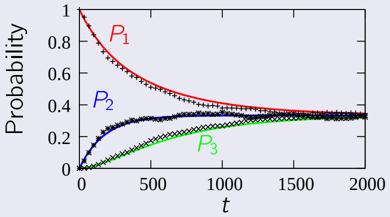

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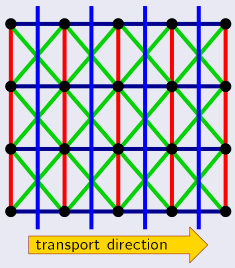
Parallel to Chains

Ballistic Transport



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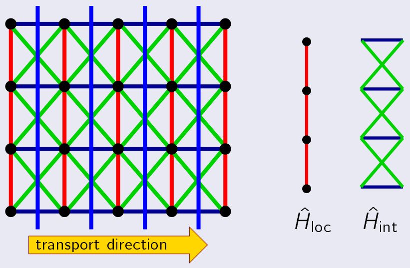

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Parallel to Chains

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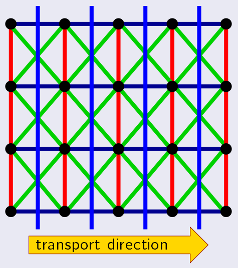

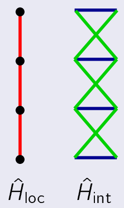

Hamiltonian

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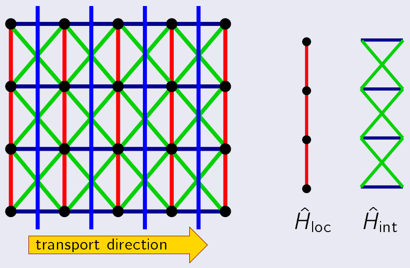
$$\hat{H}_{\text{int}} = \lambda_{\text{Heis}} \hat{H}_{\text{Heis}} + \lambda \hat{H}_{\text{rand2}}$$

Projector

$$\mathcal{P}\,\hat{\rho} = \sum_{\mu} P_{\mu} \frac{1}{n} \hat{P}_{\mu}$$

Parallel to Chains

Ballistic Transport



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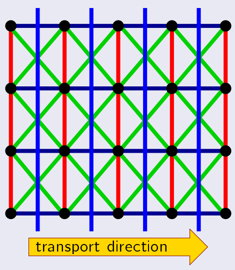
TCL master equation

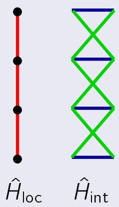
$$\frac{\mathrm{d}}{\mathrm{d}t} P_1 = -\kappa \left(P_1 - P_2 \right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t} P_{\mu} = -\kappa \left(2P_{\mu} - P_{\mu+1} - P_{\mu-1} \right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t} P_N = -\kappa \left(P_N - P_{N-1} \right)$$

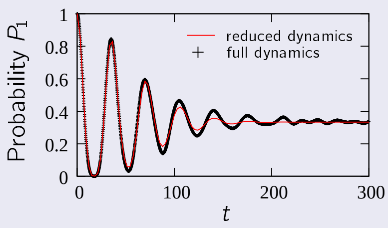
$$\kappa \propto n\lambda^2 t$$

Parallel to Chains

Ballistic Transport







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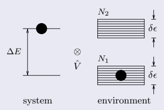
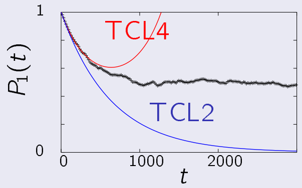
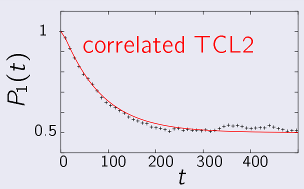
Summary

Relaxation due to Finite Environments



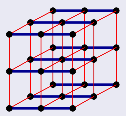


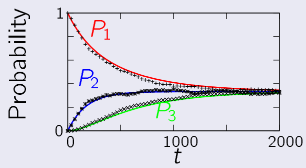
Standard Projector



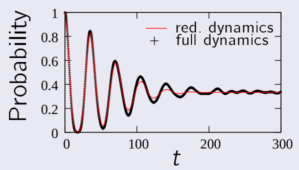
Correlated Pojector

Transport in Quantum Mechanics Using the Correlated Projector





Normal Transport



Ballistic Transport